Lecture 10: K3 Surfaces & Enrique Surfaces

Definition: A compact complex surface X is a K3 surface

Kummer, Kähler
Kodaira

if $b_1(X) = 0$ & $K_X \cong \mathcal{O}_X$.

∃ nowhere vanishing holo (2,0)-form
unique up to $\mathbb{C}^*$-scaling

$$\Downarrow$$

$$\pi_1(X) = \{1\}$$

Otherwise $\exists\ Y \longrightarrow X$ n-fold étale cover

$$K_Y \cong \mathcal{O}_Y,\quad P_2(Y) = 1,\quad \chi(\mathcal{O}_Y) = 2$$

$$2 = \chi(\mathcal{O}_Y) = \int_Y Td(Y) = n\int_X Td(X) = n\,\chi(\mathcal{O}_X) = 2n \implies n = 1$$

Hirzebruch
Riemann-Roch

Noether's formula $\chi(\mathcal{O}_X) = \frac{1}{12}(K^2 + \chi_{Top}(X))$

$$\underset{1}{h^0(\mathcal{O}_X)} - \underset{0}{h^1(\mathcal{O}_X)} + \underset{1}{h^2(\mathcal{O}_X)} = \frac{1}{12}(0 + 2 - \overset{0}{\cancel{2b_1}} + b_2) \implies b_2 = 22$$

Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & \boxed{20} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \longrightarrow \quad H^1(X, T_X) \overset{\lrcorner K_X}{\cong} H^1(X, \Omega_X)$$

20 dim'l

$H^2(K3, \mathbb{Z})$
$\shortparallel$
$$\mathbb{L}_{K3} \cong H^{\oplus 3} \oplus (-E_8)^{\oplus 2}, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

∃! $\Uparrow$ lattice theory

E_8: unique even unimodular positive definite lattice of rank 8

rank, signature + even, unimodular

Hodge diamond → rank; Hodge index theorem → signature; topology → even; Poincare duality → unimodular

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$x \in \mathbb{L}_{K3},\quad \langle x, x\rangle = \underset{\text{Thom's formula}}{Sq^{|x|}(x) = \pi^* w_2 \cup x} \underset{\text{mod } 2}{=} 0,$$

since $w_2 \equiv c_1 \pmod 2$

Theorem (Tian–Todorov) Deformation of CY are unobstructed

$\rightsquigarrow$ K3 has 20-dim'l moduli

ex. $\underline{(4) \subseteq \mathbb{P}^3}$. $\qquad (2,3) \subseteq \mathbb{P}^4$, $\qquad \underline{(2,2,2) \subseteq \mathbb{P}^5}$

dimension count $\left[\binom{4+3}{3} - 1\right] - \left[4^2 - 1\right] = 19$

Coefficents of quartics $\qquad$ PGL(4)

$\dim Gr(\mathbb{P}^2, \mathbb{P}^{20}) - \left[6^2 - 1\right]$

$20 = \left[\binom{2+5}{5} - 1\right]$ $\qquad$ PGL(6)

$\rightarrow 3 \times 18$

by adjunction formula.

+ Lefschetz hyperplane thm

Algebraic K3 has only 19-dim'l.

ex.

$$\begin{tikzcd} X \arrow[d, "2:1"] \\ \underline{\mathbb{P}^2 \supseteq (6)} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "2:1"] \\ Y = \mathbb{P}^1 \times \mathbb{P}^1 \supseteq (4,4) = B \end{tikzcd}$$

$\left[\binom{6+2}{2} - 1\right] - \left[3^2 - 1\right]$

elliptic K3

$\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(B) \xrightarrow{p} Y$

$s \in H^0(Y, \mathcal{L}^{\otimes 2})$ s.t $s^{-1}(0) = B$

$X = \{t^2 = p^* s\} \subseteq Tot(p^* \mathcal{L}^{\otimes 2})$

t : tautological section

$K_X = \pi^*(K_Y \otimes \mathcal{L})$

ex. Kummer surface

T^2 : complex 2-torus

$\circlearrowleft$

ι : involution

$x \mapsto -x$

$\rightsquigarrow$

$$\begin{tikzcd} T^2/\langle 1, \iota \rangle \\ X \arrow[u, "\pi"] \end{tikzcd}$$

has $2^4 = 16$ $\mathbb{Z}_2$-orbifold points

blow up 16 points

Kummer surface

local model

$$\begin{tikzcd} T^*\mathbb{P}^1 \arrow[d, "\pi"'] \\ \mathbb{C}^2/\mathbb{Z}_2 \end{tikzcd} \qquad \begin{tikzcd} (x, \lambda) \arrow[d, mapsto] \arrow[r, leftrightarrow] & (\overset{x^{-1}}{\overset{\parallel}{y}}, \overset{\lambda x^{-2}}{\overset{\parallel}{\mu}}) \\ (\underset{u}{\underset{\parallel}{\lambda^{\frac{1}{2}}}}, \underset{v}{\underset{\parallel}{\lambda^{\frac{1}{2}} x}}) & \end{tikzcd}$$

crepant resolution

$\uparrow$

$$\pi^* du \wedge dv = d\lambda^{\frac{1}{2}} \wedge d(\lambda^{\frac{1}{2}} x) = \frac{1}{2} d\lambda \wedge dx$$

canonical 2-form on cotangent bundle

Lemma 1: $X = K3$ surface, $D \in \mathrm{Pic}(X)$ w/ $D^2 \geq -2$
then D or $-D$ is linear equivalent to an effective divisor.

pf: Riemann-Roch $h^0(D) + h^0(-D) \geq \underset{\underset{2}{\parallel}}{\chi(\mathcal{O}_X)} + \frac{1}{2}D^2 \underset{D^2 \geq -2}{\geq} 1$ □

- $\underset{\text{has 2 components}}{\{x \in L_{K3} \otimes \mathbb{R} \mid \langle x,x \rangle > 0\}} = \underset{\text{positive cone}}{\underline{\mathcal{C}_X}} \cup \mathcal{C}'_X$ $\quad x, y \in \mathcal{C}_X \Longrightarrow \langle x,y \rangle > 0$

- Kähler cone of X

  $\{x \in \mathcal{C}_X \mid \langle x,d \rangle > 0, \ \forall$ nodal class $\underline{d}\}$

  represented by an effective (-2)-curve

- d: nodal class $\rightsquigarrow$ Picard-Lefschetz reflection

  $x \longmapsto x + \langle x,d \rangle d = s_d(x)$

  is an isometry of L_{K3} sending $\mathcal{C}_X$ to $\mathcal{C}_X$

  $s_d|_{\underset{\underset{\{\langle x,d \rangle = 0\}}{\parallel}}{H_d}} = id$, $\underline{H_d \cap \mathcal{C}_X \neq \phi}$ $\because$ signature on H_d is $(1,18)$

- Kähler cone of X is the foundamental domain of $\underset{\underset{\langle S_d \rangle \ d:\text{ nodal}}{\parallel}}{W_X}$

Theorem 1: $X = K3$ surface w/ holo $(2,0)$-form ω
Assume $d \in H^2(X, \mathbb{Z})$ s.t $\omega . d = 0$, $d^2 = 0$
then X admits an elliptic fibration.

pf: • $\underset{\underset{0}{\parallel}}{H^1(X, \mathcal{O}_X)} \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) \underset{\uparrow}{\longrightarrow} H^2(X, \mathcal{O})$

compatible w/ projection $H^2(X, \mathbb{C}) \longrightarrow H^{0,2}(X)$

$\omega . d = 0 \Longrightarrow d \in \mathrm{Pic}(X)$, say realized by $D \geq 0$

- $\exists$ nef divisor D' s.t $D'^2 = 0$.

  If D is not nef, then $\exists$ (-2)-curve C s.t $D.C < 0$

  $s_C(D) \in \overline{\mathcal{C}}_X$ and still effective

  $$0 < s_C(D).H = (D.H) + (\underset{\overset{\wedge}{0}}{D.C})(\underset{\overset{\vee}{0}}{C.H}) < D.H$$

  Therefore, after finitely many Picard-Lefschetz reflections, one reaches a nef line bundle D' w/ $D'^2 = 0$, $H^0(X, D') \neq 0$. Replace D by D'

- D is base point free.

  $$h^0(D) - h^1(D) + \cancel{h^0(-D)}^{\,0} = 2 + \cancel{\tfrac{1}{2}D^2}^{\,0} \Longrightarrow h^0(D) \geqslant 2$$

  Write $D = \underset{\text{mobile part}}{M} + \underset{\text{fixed part}}{F}$, $|M|$ has at most isolated fixed points

  $$\Downarrow$$

  M is nef

  $$\Downarrow$$

  $M.F \geqslant 0$, $M^2 \geqslant 0$

  $$\begin{tikzcd} D^2 = 0,\ D \text{ nef} \arrow[r, Rightarrow] & \underline{D.M = 0} \arrow[d] & \underline{D.F = 0} \arrow[d, Rightarrow] \\ & M^2 = M.F = 0 \arrow[r, Rightarrow] & F^2 = 0 \end{tikzcd}$$

  $F^2 = 0$ + R.R $\Longrightarrow h^0(F) \geqslant 2$ contradicts to F fixed part
  $F \neq 0$

  Therefore, $F = 0$ or D is base point free.

- Bertini theorem $\Longrightarrow$ generic elements in D are smooth but possibly disconnected.

$$\begin{tikzcd}
\varphi_{|D|}: X \arrow[rr] \arrow[dr] & & \mathbb{P}^{h^0(X,D)-1} \\
& B \arrow[ur] & \text{curve}
\end{tikzcd}$$

$\because D^2=0$ $\varphi_{|D|}$ contracts D

$\rightsquigarrow$ Stein factorization

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "\text{Connected fibres}"'] & & B \\
& \widetilde{B} \arrow[ur, "\text{finite}"'] & \\
& \text{smooth} &
\end{tikzcd}$$

Leray spectral sequence

$H^1(\widetilde{B},\mathcal{O}_{\widetilde{B}}) \hookrightarrow H^1(X,\mathcal{O}_X)=0$

$\Longrightarrow \widetilde{B}\cong\mathbb{P}^1$

Bertini theorem $\Longrightarrow$ generic fibres of $X \longrightarrow \widetilde{B}$ are smooth E of genus 1 by adjunction.

& $D\sim mE$

□

Similarly, $C\subseteq X$ effective curve w/ $C^2=2g-2$

then from the same calculation above $\Longrightarrow$ $\boxed{h^0(X,\mathcal{O}_X(C))=g+1}$

If $g\geq 1$, then

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_X(C) \arrow[r] & \mathcal{O}_C(C) \arrow[r] \arrow[d, phantom, "\cong" sloped] & 0 \\
& & & K_C &
\end{tikzcd}$$

adjunction $K_C\cong(K_X+C)|_C\cong\mathcal{O}_C(C)$

$|K_C|$ is base point free on C

$\Longrightarrow$ $\boxed{|C| \text{ is base point free}}$

$$\begin{tikzcd}
\rightsquigarrow \phi: X \arrow[r, "|C|"] & \mathbb{P}^g \\
C \arrow[r, "|K_C|"] \arrow[u, phantom, "\subseteq" sloped] & \mathbb{P}^{g-1} \arrow[u, phantom, "\subseteq" sloped]
\end{tikzcd}$$

Lemma 2: $g\geqslant 3$, then either

- ϕ birational, generic curve of $|C|$ is non-hyperelliptic.

- ϕ is $2-1$ morphism to a (possibly singular) rational surface of degree $g-1$ in $\mathbb{P}^g$. A generic curve in $|C|$ is then hyperelliptic.

pf: $g_C \geq 2$, then $|K_C|$ very ample iff X not hyperelliptic.

If C is not hyperelliptic, then $\phi|_C$ embedding $\Rightarrow$ ϕ birational.

Otherwise, C is hyperelliptic

$\Rightarrow$ generic $x \in X$, $|\phi^{-1}(\phi(x))| = 2$ or $\deg(\phi) = 2$

$C^2 = 2g-2 \Rightarrow \mathrm{Im}(\phi)$ is of degree $g-1$ in $\mathbb{P}^g$

Hyperplane sections are of the form $\underline{\phi(C')}$, $C' \in |C|$

rational

$\longrightarrow \mathbb{P}^1 \times Gr(2, g+1) \twoheadrightarrow \mathrm{Im}(\phi)$

Cut down dimension $\mathrm{Im}(\phi)$ is unirational thus rational.

□

projectic K3s $= \bigcup_{g \geq 0} \mathcal{F}_g$. elements in $\mathcal{F}_g$ admit a g-dim'l linear system $|L|$

$n(g) = \#$ of rational curves in $|L|$ for generic $X \in \mathcal{F}_g$

Theorem (Yau-Zaslow, Beauville)

$$\sum_{g \geq 0} n(g) q^g = \frac{q}{\Delta(q)}, \qquad \underline{\Delta(q) = q \prod_{n=1}^{\infty} (1+q^n)^{24}}$$

modular form of weight 12.

$$= \prod_{n=1}^{\infty} \frac{1}{(1+q^n)^{24}}$$

$$= 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + \cdots$$

Sketch of proof:

- $$\begin{tikzcd} X \arrow[r, "|L|"] & \mathbb{P}^g \arrow[r, squiggly] & \bar{J}^g\mathcal{C} \arrow[d] & \supseteq & \bar{J}C_t \\ & & {|L|} & \ni & C_t \arrow[u, squiggly] \end{tikzcd}$$

compactified relative Jacobian

$\bar{J}^g\mathcal{C}$ birat'l to $X^{[g]}$, Hilbert scheme of g points on X.

(Batyrev, Wang) Birational CYs have the same Hodge numbers.

$$\therefore \quad e(\bar{J}^g\mathcal{C}) = e(X^{[g]})$$

(Göttsche) $\sum_{g\geq 0} e(X^{[g]})\, q^g = \frac{q}{\Delta(q)}$.

- $e(X) = \underset{\text{closed}}{e(Y)} + \underset{\text{open}}{e(X \setminus Y)}$

$$e(\bar{J}C_t) = \begin{cases} 0, & C_t \text{ not rational} \\ 1, & C_t \text{ rational w/ only nodes as singularities.} \end{cases}$$

- For generic X, all the rational curves in $|L|$ are nodal.

$\square$

This motivates the development of "reduced Gromov-Witten"

$$\begin{tikzcd} \mathcal{L}_{[f]} = H^1(f^*TX) & \mathcal{L} \arrow[d] & \text{obstruction bundle} \\ & {[f] \in \mathcal{M}(C \xrightarrow{f} X, f_*[C])} & \\ T_{[f]}\mathcal{M} = H^0(f^*TX) & H_2(X,\mathbb{Z}) \arrow[u] & \end{tikzcd}$$

$$0 \to TC \to f^*TX \to N_{C/X} \to 0$$

$$0 \to \underset{\text{Aut}(C)}{H^0(TC)} \to \underset{\text{Def}(f)}{H^0(f^*TX)} \to \underset{\text{Def}(f(C)\subseteq X)}{H^0(N_{C/X})}$$

$$\to \underset{\text{Def}(C)}{H^1(TC)} \to \underset{\text{Ob}(f)}{H^1(f^*TX)} \to \underset{\text{Ob}(f(C)\subseteq X)}{H^1(N_{C/X})} \to 0$$

Original GW $= c_1(\mathcal{L}) \cap [\mathcal{M}]^{vir}$ ($\cap$ further cycle constraints)

In the case of K3 surfaces, or generally holo. symplectic mfd.

$$H^1(C, f^*TX) \xrightarrow[\simeq]{\lrcorner K_X} H^1(C, f^*\Omega_X) \longrightarrow H^1(C, \Omega_C) \xrightarrow{tr} \mathbb{C}$$

Therefore, $\mathcal{L}$ has a trivial quotient or $c_1(\mathcal{L})=0$.

Taking out the trivial quotient $\rightsquigarrow$ "reduced GW"

K3 surfaces are <u>hyperKähler</u>.

(Yau '87) X: K3 surface. given any Kähler class $[\omega]$

$\exists$ <u>Ricci-flat metric $\omega \in [\omega]$</u>.

$\Updownarrow$

$2\omega^2 = \Omega \wedge \overline{\Omega}$, after normalization of Ω.

$\Updownarrow$

Ω holo. (2,0)-form is covariant const. $\underline{\nabla}\Omega = 0$

Levi-Civita connection

$\Rightarrow$ holonomy of $X \subseteq SU(2) = Sp(1)$

i.e $\exists I, J, K$ complex structures compatible w/ g

satisfying quaternionic relations, $I^2 = J^2 = K^2 = -\mathrm{id}$

$$\underset{\underset{-JI}{\parallel}}{IJ} = K, \quad \underset{\underset{-KJ}{\parallel}}{JK} = I, \quad \underset{\underset{-IK}{\parallel}}{KI} = J$$

I, J, K arise the following way:

$$\omega(v_1, v_2) = g(Iv_1, v_2), \quad \Omega(v_1, v_2) = \underbrace{g(Jv_1, v_2)}_{\omega_J} + i\,\underbrace{g(Kv_1, v_2)}_{\omega_K}$$

Define Ω in above formula

- $\Omega(v_1, v_2) = g(Jv_1, v_2) + i\,g(Kv_1, v_2) = g(JJv_1, Jv_2) + i\,g(KKv_1, Kv_2)$

$$\parallel$$

$-\Omega(v_2, v_1) = -g(Jv_2, v_1) - i\,g(Kv_2, v_1) = -g(v_1, Jv_2) - i\,g(v_1, Kv_2)$

$\therefore$ Ω is a 2-form

- If v anti-holo. i.e $Iv = -iv$

$$\Omega(v,v') = g(KIv,v') + ig(Kv,v') = g(-iKv,v') + ig(Kv,v') = 0$$

∴ Ω holo 2-form w.r.t I

thus locally decomposable $\Longrightarrow$ $\Omega\wedge\Omega = 0$

ex. $(\mathrm{Im}\Omega + i\omega)^2 = (\mathrm{Im}\Omega)^2 - \omega^2 + 2i\,\mathrm{Im}\Omega\wedge\omega \underset{\underset{2\omega^2 = \Omega\wedge\overline{\Omega}}{\Uparrow}}{=} 0$

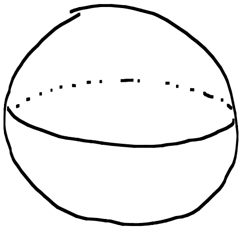
$\underset{\text{linear algebra}}{\Longrightarrow}$ $\underset{\underset{\alpha_1\wedge\alpha_2}{\shortparallel}}{\mathrm{Im}\Omega + i\omega}$ locally decomposable

thus define an almost complex structure (in this case J)

s.t $\alpha_1, \alpha_2 \in T^{1,0}$

$d(\mathrm{Im}\Omega + i\omega) \underset{\underset{\substack{d\Omega = 0\\ d\omega = 0}}{\Uparrow}}{=} 0$ $\xrightarrow[\text{theorem of Newlander–Nirenberg}]{}$ J is integrable

Actually there are an S^2-family of complex structures



$\zeta \in \mathbb{C}^* \subseteq \mathbb{P}^1$

$$J_\zeta = \frac{\sqrt{-1}(-\zeta+\bar{\zeta})I - (\zeta+\bar{\zeta})J + (1-|\zeta|^2)K}{1+|\zeta|^2}$$

$$\Omega_\zeta = \Omega + 2\zeta\sqrt{-1}\,\omega - \zeta^2\overline{\Omega}$$

$$\omega_\zeta = \frac{\sqrt{-1}(-\zeta+\bar{\zeta})\omega - (\zeta+\bar{\zeta})\omega_J + (1-|\zeta|^2)\omega_K}{1+|\zeta|^2}$$

twistor space

$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

$\mathcal{X} \underset{\text{top}}{\cong} K3 \times S^2$ complex but not symplectic

$\rightsquigarrow$ Ω_ζ glues to a section of $\Omega^2_{\mathcal{X}/\mathbb{P}^1} \otimes \mathcal{O}(2)$

$\mathbb{P}^1$ twistor line

(Lee) Given $\beta \in H_2(X,\mathbb{Z})$, $\exists!\ \zeta \in \mathbb{P}^1$ s.t.
β can be realized as holo. curves in X_ζ.

$$\beta^2 = g, \quad \int_{[M_\beta(\ominus \to X)]^{vir}} 1 = n(g)$$

X : K3 surface, ω : nowhere vanishing $(2,0)$-form
then $\omega \wedge \omega = 0$, $\omega \wedge \overline{\omega} > 0$

With $H^2(X,\mathbb{Z}) \cong \mathbb{L}_{K3}$, (X,ω) corresponds a point in

$$\Omega = \{ \underbrace{[\omega]} \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C}) \mid \omega\wedge\omega = 0,\ \omega \wedge \overline{\omega} > 0 \},$$

determined by periods $\int_A [\omega]$, $A \in H_2(X,\mathbb{Z})$

period domain of K3 surface.

called the <u>period</u> of X.

Theorem (Torelli Theorem)

$X, X' =$ K3 surfaces, $H^2(X',\mathbb{Z}) \xrightarrow{\varphi} H^2(X,\mathbb{Z})$ <u>effective Hodge isometry</u>

$\Rightarrow \varphi = f^*$, $f: X \xrightarrow{\cong} X'$ biholo.

1. preserves cut product
2. preserves Hodge decomposition
3. sends Kähler cone to Kähler cone

(weak Torelli) $H^2(X',\mathbb{Z}) \xrightarrow{\cong} H^2(X,\mathbb{Z})$
then $X \cong X'$

Given a family of K3 surface, a <u>marking</u> is an isomorphism

$$\begin{tikzcd} X_s \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ s \arrow[r, phantom, "\in"] & S \end{tikzcd}$$

$\alpha : R^2\pi_*\mathbb{Z} \cong \underline{\mathbb{L}_{K3}}$ on S

constant lattice

fibrewise $H^2(X_s) \cong \mathbb{L}_{K3}$

$S \ni s_0$ reference

Step 1. Local Torelli Theorem

$$\begin{tikzcd}
T_sS \arrow[r, "\text{Kodaira-Spencer}"] \arrow[d] & H^1(X_s, TX_s) \arrow[d, phantom, sloped, "\cong"] \arrow[d, phantom, "\llcorner K_{X_s}"', shift left=6ex] \\
T_{\omega_s}\Omega \arrow[d, phantom, sloped, "\cong"] & H^1(X_s, \Omega_{X_s}) \arrow[d, phantom, sloped, "\cong"] \arrow[d, phantom, "\text{Dolbeault isom.}"', shift left=10ex] \\
\mathrm{Hom}(\mathbb{C}\omega_s, H^{1,1}(X_s)) \arrow[r, phantom, "\cong"] & H^{1,1}(X_s)
\end{tikzcd}$$

$T_V Gr(n,k) \cong \mathrm{Hom}(V, W/V)$

Ω further restricts $\omega^2 = 0$

⇝ local Torelli theorem

Given a family of marked K3 surface $\begin{matrix}\mathcal{X}\\ \downarrow \\ S\end{matrix} \ni s_0$ reference point

the periods determined the isomorphism classes of fibres near $s_0 \in S$.

Step 2. Torelli Theorem for Projective Kummer Surfaces

X: projective Kummer, $X' =$ K3 surface

If $H^2(X, \mathbb{Z}) \xrightarrow{\phi} H^2(X', \mathbb{Z})$ effective Hodge isometry.

then $\phi = f^*$, $X' \xrightarrow{f} X$ biholo.

- $\mathcal{O}_X(\sum_{i=1}^{16} \underbrace{C_i}_{(-2)\text{-curve}})$ is 2-divisible ⇝ $\exists\ \tilde{X}$, $\tilde{C}_i$, $\tilde{C}_i^2 = -1$

$$\begin{tikzcd}
\exists\ \tilde{X} \arrow[d] & \tilde{C}_i ,\ \tilde{C}_i^2 = -1 \arrow[d] \\
X \text{ ramfield on} & C_i
\end{tikzcd}$$

$\Rightarrow X'$ is also Kummer.

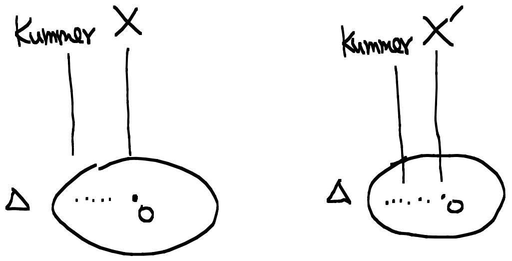
- Prove $\exists$ biholo. on the corresponding tori.

Step 3. Density Theorem

Periods of projective Kummer surfaces are dense in Ω.

As a Corollary, all K3s are diffeomorphic.

Step 4. Given X, X' K3 surface w/ a Hodge isometry $H^2(X',\mathbb{Z}) \xrightarrow{\varphi} H^2(X,\mathbb{Z})$.



Kuranishi family w/ markings $p: \mathcal{X} \to \Delta$

$p': \mathcal{X}' \to \Delta'$

- Construct the Kuranishi family near X, X'

local Torelli theorem $\Delta \cong \Delta'$

w/ $R^2 p'_* \mathbb{Z}_{\mathcal{X}'} \xrightarrow[\cong]{\Phi} R^2 p_* \mathbb{Z}_{\mathcal{X}}$ extending φ.

- By density theorem & Torelli theorem for projective Kummer surfaces

$\exists\, s_n \longrightarrow 0 \in \Delta$

s.t. $\underset{\Phi(s_n)}{\underset{\parallel}{\Phi_n}}: \underset{X_n}{p^{-1}(s_n)} \cong \underset{X_n'}{\underset{\parallel}{p'^{-1}(s_n)}}$

View Φ_n as a correspondence in $X_n \times X_n'$

$\Longrightarrow$ $\Phi_n \longrightarrow \underline{\Phi_0} \in X_0 \times X_0'$ (in Barlet topology)

purely 2-dim'l analytic cycle represents element in $H^4(X_0 \times X_0', \mathbb{Z})$.

The correspondence induces isomorphism $H^k(X_0,\mathbb{Z}) \longrightarrow H^k(X_0',\mathbb{Z})$, $\forall k$

$\Rightarrow$ $\Phi_0 \xrightarrow{p_2} X_0'$ of degree 1

$k=4$

$\Phi_0 = \Delta + \Delta_1$. $p_2: \Delta \longrightarrow X'$ birational

$p_2(\Delta_1) \subseteq X'$ lower dimension

$k=2$, $\underset{\Delta + C}{\overset{\Phi_0}{\shortparallel}} \xrightarrow{p_1} X_0$, thus $\Delta \xrightarrow{p_1} X$ birational

Otherwise $\Delta_1 \longrightarrow X$ birational

$\Phi_0(H^2(X,\mathbb{Z})) \subseteq NS(X_0') \subsetneq H^2(X',\mathbb{Z})$ $\longrightarrow\!\!\!\!\!\!\times$

In particular, X_0 & X_0' birational

X_0, X_0' minimal non-ruled $\Longrightarrow$ $X_0 \cong X_0'$ weak Torelli theorem

Δ isom, $\Delta' = \sum_i C_i \times D_i$

If moreover, Φ_0 induces Hodge isometry.

On $\underset{x}{\overset{}{H^2(X,\mathbb{Z})}}$, $\Phi_0(x) = x + \sum_i \langle C_i, x\rangle d_i$, C_i, d_i effective class

$$0 = \langle \Phi_0(x), \Phi_0(x)\rangle - \langle x, x\rangle$$

$$= \langle \underbrace{\Phi_0(x)+x}_{\in\, \mathcal{C}_X^+}, \underbrace{\Phi_0(x)-x}_{= \sum_i \langle C_i, x\rangle d_i} \rangle > 0 \quad \longrightarrow\!\!\!\!\!\!\times$$

$\mathcal{C}_X^+ \supseteq$ Kähler cone

$\mathcal{C}_X^+ = \{ x \in \mathcal{C}_X \mid \langle x, d\rangle \in \mathbb{R}_+ \}$

positive cone $\quad d = (-2)$-class

It worth noticing that the moduli space of marked K3 surfaces is fine but not separated.

ex. Consider a 1-parameter family of K3 surface

$$\begin{tikzcd} \mathcal{X} \arrow[d] & \supseteq & X_0 \arrow[d] \\ \Delta & \ni & 0 \end{tikzcd}$$

X_0 has an ordinary double point

One can have two different small resolutions on $\mathcal{X}$, modifying X_0.

connected by an Atiyah flop

the exceptional curve is (-2)-curve in the fibre over 0.

The two distinct markings differ by a Picard-Lefschetz reflection of the (-2)-curve.